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ESTIMATION OF AREAS ON SOME SURFACES DEFINED BY THE SYSTEM OF EQUATIONS

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Abstract. In this article, we consider some metric questions on Manifolds defined by the system of equations. We obtain estimates in terms of products of singular numbers of some matrices defined by taking all partial derivatives of entries of a given matrix of the same order.

Keywords: differentiable function, surface integral, system of equations, Jacobi Matrix, surface integrals.

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1. Introduction

Many problems of Analysis and Applied Mathematics lead to the investigations of some metric questions in differentiable manifolds [1, 3, 10]. Often these Manifolds are given by a system of equations in formulation of which involve differentiable functions. Such systems of equations arose in the question on the convergence exponent of the special integral of Terry's problem ([2, 5-9]). In this paper we estimate areas of some surfaces defined by the system of equations. Such a problem arises when one considers the question on estimations of oscillatory integrals ([2, 3, 5-9]).

2. Auxiliary lemmas

To prove our results we will use some auxiliary statements. These statements are considered in the lemmas below.

Lemma 2.1. *Let in a bounded Jordan domain Ω of n-dimensional space R^n some continuous function $f(\bar{x}) = f(x_1, ..., x_n)$ and continuously-differentiable functions $f_j(\bar{x}) = f_j(x_1, ..., x_n)$ be given, with $j = 1, ..., r$, $r < n$, such that the Jacobi Matrix*

$$\frac{\partial(f_1, ..., f_r)}{\partial(x_1, ..., x_n)},$$

has everywhere in Ω a maximal rank. Let, further $\bar{\xi}_0 = (\xi_1^0, ..., \xi_r^0)$ be an inner point for the image of a map $\bar{x} \mapsto (f_1, ..., f_r)$ and $\bar{x}_0$ is a point of the domain Ω such that

$$f_1(\bar{x}_0) = \xi_1^0, ..., f_r(\bar{x}_0) = \xi_r^0.$$

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Then, everywhere in a neighborhood of the point $\bar{\xi}_0$ the following equality

$$\frac{\partial^r}{\partial\xi_1\cdots\partial\xi_r}\int\limits_{\Omega(\bar{\xi})} f(\bar{x})d\bar{x} = \int\limits_{M(\bar{\xi})} f(\bar{x})\frac{ds}{\sqrt{G}},$$

holds, where $\Omega(\bar{\xi})$ is a subdomain of Ω defined by the system of inequalities $f_j(\bar{x}) \le \xi_j$, $M(\bar{\xi})$ is a surface defined by the system of equations $f_j(\bar{x}) = \xi_j (j = 1, ..., r)$, and G is Gram determinant of gradients of the functions $f_j(\bar{x})$, i. e. $G = |(\nabla f_i, \nabla f_i)|$ (see [12, p.92]); at the right hand side the surface integral stands (see [4]).

Consequence 2.1. *Let the conditions of the lemma 2.1 be satisfied. Then, we have*

$$\int\limits_{\Omega} f(\bar{x})d\bar{x} = \int\limits_{m_1}^{M_1}\cdots\int\limits_{m_r}^{M_r} du_1\cdots du_r \int\limits_{M} f(\bar{x})\frac{ds}{\sqrt{G}},$$

where m_j and M_j, correspondingly, minimal and maximal values of the functions $f_j(\bar{x})$, $m_j \le f_j(\bar{x}) \le M_j$. $M = M(\bar{u})$ is a surface in Ω defined by the system of equations, $f_j = u_j, j = 1, ..., r$, and G is a Gram determinant of the functions defined M].

Lemma 2.2. *Let, under conditions of the lemma 1, the equalities $\xi_1^0 = ... = \xi_r^0 = 0$ be satisfied and the surface M be defined by the system of equations*

$$f_1(x_1, ..., x_n) = 0,$$

$$\cdots$$

$$f_r(x_1, ..., x_n) = 0,$$

moreover the functions $f_j(\bar{x})$ are continuously differentiable in some domain Ω_0, including the domain Ω. Let $G = G(\bar{x})$ be a Gram determinant of gradients of the functions $f_j(\bar{x})$ distinct from zero in Ω. Let, further, transformation of coordinates $\bar{x} = \bar{x}(\bar{\xi})$ be one to one mapping of some domain Ω' to Ω with a non-singular Jacobi matrix

$$Q = Q(\bar{\xi}) = \left(\frac{\partial x_i}{\partial \xi_j}\right)_{1\le i,j\le n},$$

having continuous in Ω entries. Then, for any continuous function $f(\bar{x})$ in Ω we have

$$\int\limits_{M} f(\bar{x})\frac{ds}{\sqrt{G}} = \int\limits_{M'} |\det Q|\, f(\bar{x}(\bar{\xi}))\frac{d\sigma}{\sqrt{G'}}, G' = \det(JQ\cdot {}^tQ^tJ),$$

where M' is a pre-image of the surface M in this transformation $d\sigma$ is a surface element in coordinates $\bar{\xi}$, J is a Jacobi matrix of the system of functions $f_j(\bar{x})$:

$$J = \frac{\partial(f_1, ..., f_r)}{\partial(x_1, ..., x_n)}$$

Proof of these statements are given in [7].

3. Main results

Let Ω be a bounded closed domain in n-dimensional spaceR^n, $n > 1$, R is a set of real numbers. Let's assume that in Ω, r dimensional surface is defined by the system of equations

$$f_j(\bar{x}) = 0, j = 1, ..., n - r,\ \ 0 \le r \le n, \qquad (1)$$

with differentiable functions on the left side, and the Jacobi matrix

$$J = J(\bar{x}) = \left(\frac{\partial f_j}{\partial x_i}\right), i = 1, ..., n; j = 1, ...n - r,$$

has, everywhere in Ω, the maximum rank.

Let further, $A_0 = A_0(\bar{x})$ be other matrix function written down in a view

$$A_0 = A_0(\bar{x}) = \|f_{ij}(\bar{x})\|, 1 \le i \le r, 1 \le j \le m,$$

with differentiable entries. Writing entries of columns of the matrix A_0 at a line

$$f_{11}(\bar{x}), ..., f_{r1}(\bar{x}), f_{12}(\bar{x}), ..., f_{r2}(\bar{x}), ..., f_{1m}(\bar{x}), ..., f_{rm}(\bar{x}),$$

take transposed Jacobi matrix of this system of functions, designating it as $A_1 = A_0'(\bar{x})$:

$$A_1 = A_1 = A_0'(\bar{x}) = (\bar{x}) = \left\| \begin{array}{ccccccc} \frac{\partial f_{11}}{\partial x_1} & \dots & \frac{\partial f_{r1}}{\partial x_1} & \dots & \frac{\partial f_{1m}}{\partial x_1} & \dots & \frac{\partial f_{rm}}{\partial x_1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{\partial f_{11}}{\partial x_n} & \dots & \frac{\partial f_{r1}}{\partial x_n} & \dots & \frac{\partial f_{1m}}{\partial x_n} & \dots & \frac{\partial f_{rm}}{\partial x_n} \end{array} \right\|.$$

Then, entries of columns of this matrix, as above, arranged in a line, and taking the transposed Jacobi Matrix $A_2 = A_2(\bar{x}) = A_1'(\bar{x})$ of the received system of functions, and we will continue this procedure while does not receive a matrix $A_k(\bar{x}) = A_{k-1}'(\bar{x})$ for this $k \ge 1$. The matrix defined by such way consists of all partial derivatives of one and the same order k of entries of the matrix $A_0 = A_0(\bar{x})$ and has the size $n \times n^{k-1}rm$. Let's assume that $A_j(\bar{x})$ has in Ω a maximal rank being equal to n. Let's designate by $G_j(\bar{x})$ the product of the last (smallest) r singular numbers of the matrix $A_j(\bar{x}), j = 0, ..., k$. We put

$$E = E(H) = \{\bar{x} \in \Omega | G_0(\bar{x}) \le H\}, H > 0.$$

If $\phi_{ik}(\bar{x})$ denotes entries of the matrix $A_j(\bar{x})$, we will accept the following designations

$$L_j(\bar{x}) = \left(\sum_{i,k} |\phi_{ik}|^2\right)^{1/2},$$

$$L = \max_j \max_{\bar{x} \in \Omega} L_j(\bar{x}), G_j = G_j(\bar{x}).$$

Below we will prove the theorem, allowing to estimate an area of E in terms of singular numbers of the matrix A_1.

For the formulation and proof of our statements we need to dissect the domain Ω into such parts in each of which the system (1) allows a one-valued solvability. These parts are defined by the maximal minors of the matrix J of the system. Let's dissect Ω into no more than $t = C_n^{n-r}$ sub domains Ω_ν intersecting each with another, at most, by parts of boundaries only. In each of these subdomains one of the minors of the Jacobi matrix has the maximum of modulus among all minors. We assume that *each subdomain Ω_ν is a closed set, and can be represented as a union of finite number of simply connected closed domains, as a set of solutions of a system of inequalities in Ω.* Therefore, each subdomain Ω_ν is represented in the form $\Omega_\nu = \bigcup_{c \le T_0} \Omega(\nu, c)$, where $\Omega(\nu, c)$ is a simply connected subdomain.

Let's consider one of subdomains $\Omega(\nu, c)$. From our assumption it follows that in any neighbourhood of a given solution of the system (1) the system allows one-valued solubility with respect to *one and the same variables.* So, the domain $\Omega(\nu, c)$ can be dissected into no more than f subdomains $\Delta_\mu, \mu = 1, ..., f, f \le F$, in each of which the system (1) allows one-valued solvability with respect to $n - r$ variables. Let $\bar{\xi} = (\xi_1, ..., \xi_r)$ be a vector composed of independent variables. Then, it is possible to present each variable x_i as a function $x_i = x_i(\bar{\xi})$

of independent variables. Let's designate $A_0(\bar{\xi})$ the matrix received by replacing in the matrix $A_0(\bar{x})$ independent variables x_i by the functions $x_i = x_i(\bar{\xi})$. So, we consider the Matrix function A_0 as a matrix of $\bar{\xi}$. Let's designate $G_{(1)}$ the minimal value of Gram determinant of gradients of the entries of the matrix $A_0(\bar{\xi})$ (here the differentiation is taken with respect to $\bar{\xi}$, i.e. $G_{(1)} = \min\det\left(A_{1\bar{\xi}} \cdot {}^t A_{1\bar{\xi}}\right)$ (we will notice that the minimum is taken over all c and ν and, therefore, depends on $\bar{\xi}$). So, the matrix $A_{1\bar{\xi}}$ is a matrix of the size $r \times rm$ received from A_0 by differentiation with respect to $\bar{\xi}$, i. e. $A_{1\bar{\xi}} = A_0'(\bar{\xi})$. Thus, the matrix being considered as a matrix of $\bar{\xi}$ differs from $A_{1\bar{\xi}}$. Further, for the positive number a we designate $h(a) = a + a^{-1}$. It is obvious that $a \leq h(a), h(a^{-1}) = h(a)$ and $h(ab) \leq h(a)h(b)$, for $a, b > 0$.

Theorem 3.1. *Let Π_H be the part of the surface (1) included in $E(H)$ and $G_{(1)} > 0$. Suppose that the Jacobi matrix of the system of entries of some column of A_0 sets up a minor of A_1 with maximal modulus. Then for the $n-r$-dimensional volume (briefly area) $\mu(\Pi_H)$ we have an estimation*

$$\mu\left(\Pi_H\right) \leq FT_0 \cdot 2^{r+3} r^{3r} c_0^2 \begin{pmatrix} nr \\ r \end{pmatrix}^{1/2} \begin{pmatrix} n \\ r \end{pmatrix}^{3/2} H \cdot G_{(1)}^{-1} \cdot \tilde{\wp}^r,$$

$$\tilde{\wp} = r^2 \log\left\{h\left(G_{(1)}\right) h\left(H\right) h\left(L\right)\right\}, c_0 = \pi^{-r/2}\Gamma\left(1 + r/2\right).$$

Proof. $\mu\left(\Pi_H\right)$ it is possible to present as a following surface integral

$$\mu\left(\Pi_H\right) = \int\limits_{\Pi\cap E(H)} ds,$$

where Π designates the surface of solutions of the system (1). Let among the parts $\Delta_\mu, \mu = 1, ..., t, t \leq F$ of the surface Π the part Δ_1 be the maximal area and

$$J_1 = \frac{D(f_1, ...f_{n-r})}{D(x_{n-r+1}, ..., x_n)},$$

designates the corresponding maximal minor. Let Π_0 be the area of the part Δ_1. Then, we receive the following estimation

$$\mu\left(\Pi_H\right) \leq F\Pi_0.$$

Now estimate Π_0.

$$\Pi_0 \leq \int\limits_{\Pi_1'} \frac{\sqrt{\Sigma J_i^2}}{|J_1|} d\xi_1...d\xi_r$$

$$\leq \left(C_n^r\right)^{1/2} \int\limits_{\Pi_1'} d\xi_1...d\xi_r, \qquad (2)$$

where Π_1' denotes a domain of changing independent variables $\xi_1, ..., \xi_r$ parameterizing the part Π_1'. It is well-known that (see [11, p.74]) $G = \sum\limits_{1\leq j\leq l} (M_j)^2$, where $M_j, 1 \leq j \leq l$ denotes modules of different minors of the Jacobi matrix of a maximal rank. The number of these minors is equal to $l = C_n^r$. Then from this equality we have:

$$|\det D_0|^2 \leq \det\left(A_0 \cdot {}^t A_0\right) \leq H^2,$$

and D_0 means the submatrix of A_0 containing such entries gradient columns of which coincides with one of minors having maximal absolute values (i. e. with the minor J_1). Therefore, according to the assumption

$$|detJ_1|^2 \geq (C_{nr}^r)^{-1} \det\left(A_1 \cdot {}^tA_1\right) \geq (C_{nr}^r)^{-1}\left(G_{(1)}\right)^2. \qquad (3)$$

Then, the integral on the right part (2) does not exceed:

$$\int\limits_{\Pi_1', |detD_0| \leq H} d\xi_1 \cdots d\xi_r \leq \sum_{j=1}^{\infty} E_j,$$

where

$$E_j = \int\limits_{2^{-j}H \leq |detD_0| \leq 2^{j-1}H} d\xi_1 \cdots d\xi_r.$$

Let $\rho_1 = \rho_1(\bar{\xi}), ..., \rho_r = \rho_r(\bar{\xi})$ where $\bar{\xi} = (\xi_1, ..., \xi_r)$, be the singular numbers of a matrix D_0 $\rho_1 \leq \cdots \leq \rho_r$. Then, from the inequality

$$2^{-j}H \leq \rho_1 \cdots \rho_r \leq \rho_r \rho_1^{r-1},$$

we deduce

$$\rho_r \geq 2^{-j}\rho_1^{1-r}H,$$

Writing $D_0 = (d_{ij})$ we have:

$$\rho_1^2 \leq \rho_1^2 + \cdots + \rho_r^2 \leq \sum_{i,j} d_{ij}^2 \leq L^2.$$

In accordance with Shur's lemma [4, p.288]. Therefore,

$$\rho_r \geq 2^{-j}HL^{1-r}. \qquad (4)$$

Let's estimate now $E_j, j = 1, 2, ...$ We have:

$$\frac{E_j}{2^{1-j}H} \leq \int\limits_{2^{-j}H \leq |detD_0| \leq 2^{j-1}H} \frac{d\bar{\xi}}{|detD_0|}$$

$$\leq c_0 \int\limits_{2^{-j}H \leq |\det D_0| \leq 2^{j-1}H} d\bar{\xi} \int\limits_{\|D_0\bar{\alpha}\| \leq 1} d\bar{\alpha}), \qquad (5)$$

where $c_0 = \pi^{-r/2}\Gamma(1 + r/2)$. Further, from an inequality

$$1 \geq \| D_0\bar{\alpha} \|^2 = ({}^tD_0 \cdot D_0\bar{\alpha}, \bar{\alpha})$$

$$\geq \rho_r^2 \| \bar{\alpha}) \|^2 \geq \rho_r^2 |\alpha_i|^2; \| \bar{\alpha} \|^2 = \sum_{i=1}^{r} |\alpha_i|^2, \qquad (6)$$

for all i due to (4), we get the following bound

$$|\alpha_i| \leq \left(\sum_{i=1}^{r}\right)^{1/2} \leq \rho_r^{-1} \leq 2^j H^{-1} L^{r-1}, \qquad (7)$$

for the variables in internal integral in (5). Let's enter into consideration a ball:

$$K = \{\bar{\alpha} | \ \| \bar{\alpha} \| \leq 2^j H^{-1} L^{r-1}.\} \qquad (8)$$

The relation (5) it is possible to represent (designating the domain defined by the condition $2^{-j}H \leq |detD_0| \leq H^{1-j}H$ as τ) as follows:

$$\frac{E_j}{2^{1-j}H} \leq c_0 \int\limits_{\tau} d\bar{\xi} \int\limits_{K, \|D_0\bar{\alpha}\| \leq 1} d\bar{\alpha} = c_0 \int\limits_{\bar{\alpha} \in K} d\bar{\alpha} \int\limits_{\bar{\xi} \in \tau, \|D_0\bar{\alpha}\| \leq 1} d\bar{\xi}. \qquad (9)$$

Remove from the ball K all of strips $K_m(m = 1, ..., r)$ defined by conditions

$$|\alpha_m| \leq \left(G_{(1)}\right)^{-1} r^{-1} 2^{j(1-r)} H^r L^{-(1-r)^2}, \tag{10}$$

$$|\alpha_i| \leq 2^j H^{-1} L^{r-1}, i \neq m. \tag{11}$$

Let's designate $K_0 = \bigcup_{m=1}^r K_m$ and estimate the measure K_0 trivially:

$$\mu(K_0) \leq c_0 H \left(G_{(1)}\right)^{-1}. \tag{12}$$

On the right part of (9), we will dissect the multiple integral into two integrals. The first it ems will be determined by the condition $\bar{\alpha} \in K_0$, and the second by the condition $\bar{\alpha} \in K \backslash K_0$. Estimate the first integral trivially using the found estimation:

$$\int\limits_{\bar{\alpha} \in K_0} d\bar{\alpha} \int\limits_{\bar{\xi} \in \tau, \|D_0 \bar{\alpha}\| \leq 1} d\bar{\xi} \leq c_0 H G_{(1)}^{-1} \tag{13}$$

To estimate the integral over $K \backslash K_0$ in the inner integral we will apply an exchange of variables. For any fixed $\bar{\alpha} \in K \backslash K_0$, we put

$$\bar{\eta} = D_0(\bar{\xi}')\bar{\alpha}.$$

Formally, the Jacobi matrix of this exchange of variables is equal to the inverse of the matrix

$$J = \frac{\partial(\eta_1, ..., \eta_r)}{\partial(\xi_1, ..., \xi_r)} = \left(\frac{\partial D_0}{\partial \xi_1} \bar{\alpha} \cdots \frac{\partial D_0}{\partial \xi_r} \bar{\alpha} \right)$$

and $\partial({}^t D_0)/\partial \xi_j$ designates a matrix received by differentiation of all entries of the matrix $D_0 = D_0(\bar{\xi})$ with respect to the variable ξ_j:

$$\int\limits_{\bar{\alpha} \in K \backslash K_0} d\bar{\alpha} \int\limits_{\bar{\xi} \in \tau, \|D_0 \bar{\alpha}\| \leq 1} d\bar{\xi} = \int\limits_{\bar{\alpha} \in K \backslash K_0} d\bar{\alpha} \int\limits_{\bar{\xi} \in \tau, \|\eta\| \leq 1} |J|^{-1} d\bar{\eta}. (*)$$

For every η we will designate by $\tau(\bar{\eta})$ a subset in $K \backslash K_0$ all of points with the constraint $\| D_0(\bar{\xi})\alpha \| \leq 1$. Changing orders of integrations in the last integral, we receive an inequality:

$$\int\limits_{\bar{\alpha} \in K \backslash K_0} d\bar{\alpha} \int\limits_{\bar{\xi} \in \tau, \|D_0 \bar{\alpha}\| \leq 1} d\bar{\xi} \leq \int\limits_{\|\eta\| \leq 1} d\bar{\eta} \int\limits_{\tau(\bar{\eta})} |J|^{-1} d\bar{\alpha},$$

extending the integration to all of the specified η. We need the proof of the relation (*). At first we note that the left hand side of the equality is finite. It is clear that the set of solutions of the equation $|J| = 0$ is closed Jordan set in $(K \backslash K_0) \times \tau$. Suppose that such a Jordan set contains an open set V. Then, for every point $(\bar{\alpha}, \bar{\xi}') \in V, \bar{\alpha} \in K \backslash K_0, \bar{\xi}' \in \tau$ there will be found such a neighborhoods of the points $\bar{\alpha} \in V', \bar{\xi}' \in \tau'$ that $V' \times \tau' \subset (K \backslash K_0) \times \tau$. Since the set of solution is a Jordan set, to prove our statement it is enough to establish that the set of points where $|J| = 0$ can't contain points $\bar{\alpha} \in \overline{K \backslash K_0}$ with its open neighborhood for all points $\bar{\xi}' \in \tau$ with add a comma their subset of zero Jordan measure. For establishing the last conclusion note that the matrix ${}^t J$ could be written as

$${}^t J = \begin{pmatrix} {}^t\bar{\alpha} \cdot \frac{\partial^t D_0}{\partial \xi_1} \\ \cdots \\ {}^t\bar{\alpha} \cdot \frac{\partial^t D_0}{\partial \xi_r} \end{pmatrix}.$$

Suppose now, in contrary, that the set of solutions of the equation $|J| = 0$ contains an open subset in $K \backslash K_0$. Then we can differentiate an identity $|J| = 0$ in this open set with respect to the components of the vector $\bar{\alpha}$. Differentiating the determinant with respect to the variable α_j we get a sum of $j!$ determinants, which are equal one to another. Each of these determinants is a Jacobian of the system of functions of j-th row of the matrix $D_0 = D_0(\bar{\xi})$. We have received a contradiction in consent with conditions of the theorem. So, our supposition is not true. Therefore the equality $|J| = 0$ is not satisfied in an open subset in $K \backslash K_0$ for each point $\bar{\xi}' \in \tau$, except of the points of a set of zero Jordan measure. Since the set of points in $(K \backslash K_0) \times \tau$ with the condition $|J| = 0$ has zero Jordan measure, then we can overlap this set by a union of open cubes with arbitrarily small total measures. The left side of the equality (*) being taken over such a covering also is small. We must prove uniform estimation for closed complement of such covering, non-dependent of it, then we have proven (*) using an improper meaning.

Now we pass to the estimation of the integral on the right side of the equality (*). For every $\bar{\eta}$ denoted by $\tau(\bar{\eta})$ the subset in $K \backslash K_0$ for all points of which the inequality $\left\| {}^t D_0 \bar{\alpha} \right\| \leq 1$ is satisfied. Consider the matrix ${}^t J$ as a matrix of linear map which puts in correspondence to every vector $\bar{\beta} \in R^r$ a vector ${}^t J \bar{\beta}$. This map is bilinear which we write as a map $\Phi : \left(\bar{\alpha}, \bar{\beta}\right) \mapsto {}^t J \bar{\beta}$. For every point $\left(\bar{\alpha}, \bar{\beta}\right) \in R^{2r}$ the equality $\Phi\left(\bar{\alpha}, \bar{\beta}\right) = D_1(\bar{\alpha} \otimes \bar{\beta})$ is satisfied, where for the vectors ${}^t\bar{\alpha} = (\alpha_1, ..., \alpha_r)$ and ${}^t\bar{\beta} = (\beta_1, ..., \beta_r)$ the symbol ${}^t(\bar{\alpha} \otimes \bar{\beta})$ will denote a direct (Cartesian) product $(\alpha_1\beta_1, ..., \alpha_1\beta_r, ..., \alpha_r\beta_1, ..., \alpha_r\beta_r)$. Writing $|J|^{-1}$ as an integral as above, from (*) we get:

$$\int\limits_{\|\bar{\eta}\| \leq 1} d\bar{\eta} \int\limits_{\tau(\bar{\eta})} |J|^{-1} \, d\bar{\alpha} = c_0 \int\limits_{\|\bar{\eta}\| \leq 1} d\bar{\eta} \int\limits_{\tau(\bar{\eta})} d\bar{\alpha} \int\limits_{\left\|D_1(\bar{\alpha} \otimes \bar{\beta})\right\| \leq 1} d\bar{\beta}. \tag{14}$$

Consider now an inner multiple integral over $\bar{\alpha}$ and $\bar{\beta}$:

$$\int\limits_{\tau(\bar{\eta}), \left\|D_1(\bar{\alpha} \otimes \bar{\beta})\right\| \leq 1} d\bar{\alpha} d\bar{\beta}. \tag{15}$$

Let the singular value decomposition of the matrix D_1 has a view $D_1 = Q\Sigma T$, where Q and T are orthogonal matrices of orders, correspondingly, r, r^2 and Σ is a block matrix of a view (Σ_1, Σ_2), with diagonal matrix Σ_1 with diagonal entries composed of singular numbers $\sigma_1, ..., \sigma_r$ of the matrix D_1, and zero matrix Σ_2. Note that columns of the matrix D_1 can be placed in Σ with any order in consent with replacement of columns of the matrices T and Q. Consider the integral (15) and make an exchange of variables $t_i = \alpha_i\beta_i$, $i = 1, ..., r$. Before application of the Lemma 1 we conduct the following reasoning. From the made exchange of variables we find: $\beta_i = t_i\alpha_i^{-1}$, which we write conditionally as $\bar{\beta} = \bar{t}\bar{\alpha}^{-1}$. Then we have

$$D_1(\bar{\alpha} \otimes \bar{\beta}) = D_1\left(\bar{\alpha} \otimes \bar{t}\bar{\alpha}^{-1}\right).$$

Counting the inequalities (10-11), for all points from $K \backslash K_0$, and all i the following inequalities are satisfied

$$\left(G_{(1)}\right)^{-1} r^{-1} 2^{j(1-r)} H^r L^{-(1-r)^2} \leq \alpha_i \leq 2^j H^{-1} L^{r-1}. \tag{16}$$

Under conditions of the theorem 1 we have $G_1(\bar{\xi}') \geq G_{(1)}$. Now we apply the lemma 2 performing exchange of variables:

$$\int\limits_{\tau(\bar{\eta})} d\bar{\alpha} \int\limits_{\|D_1(\bar{\alpha} \otimes \bar{t}\bar{\alpha}^{-1})\|} d\bar{\beta}$$

$$= \int\limits_{d} \bar{t} \int\limits_{t_i = \alpha_i\beta_i, \left\|D_1(t\bar{\beta}^{-1} \otimes \bar{\beta})\right\|} \frac{ds}{\sqrt{\alpha_1^2 + \beta_1^2} \cdots \sqrt{\alpha_r^2 + \beta_r^2}}. \tag{17}$$

Transforming the surface integral into the multiple integral we get:

$$\int\limits_{t_i = \alpha_i\beta_i, \left\|D_1(t\bar{\beta}^{-1} \otimes \bar{\beta})\right\|} \frac{ds}{\sqrt{\alpha_1^2 + \beta_1^2} \cdots \sqrt{\alpha_r^2 + \beta_r^2}} \leq \int \frac{d\alpha_1 \cdots d\alpha_r}{\alpha_1 \cdots \alpha_r},$$

recalling that the bounds of variation of the variables α_i are defined by the inequalities (16). To estimate from above the multiple integrals on the right hand side of (17) we change the order of integration:

$$\int \frac{d\alpha_1 \cdots d\alpha_r}{\alpha_1 \cdots \alpha_r} \int\limits_{\|D_1(\bar{\alpha}\otimes \bar{t}\bar{\alpha}^{-1})\|\leq 1} dt_1 \cdots dt_r.$$

Consider now in R^{r^2} a manifold of dimension *2r*:

$$t_{11} = \alpha_1\beta_1, ..., t_{1,r} = \alpha_1\beta_r, ..., t_{r,1} = \alpha_r\beta_1, ..., t_{rr} = \alpha_r\beta_r.$$

Inner integral in the last multiple integral can be represented as a surface integral over the considered line manifold $\bar{\alpha} \otimes \bar{t}\bar{\alpha}^{-1}$ in R^{r^2} of dimension r. The r dimensional element of the volume can be represented as

$$\left|U \cdot {}^tU\right| dt_1 \cdots dt_r,$$

where

$${}^tU = \begin{pmatrix} 1 & 0 & \cdots & 0 & \alpha_2\alpha_1^{-1} & 0 & \cdots & 0 & \cdots \\ 0 & \alpha_1\alpha_2^{-1} & \cdots & 0 & 0 & 1 & \cdots & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

So, we have

$$\int\limits_{\|D_1(\bar{\alpha}\otimes \bar{t}\bar{\alpha}^{-1})\|\leq 1} dt_1 \cdots dt_r = \int\limits_{\|D_1\bar{x}\|\leq 1} \frac{ds}{|U \cdot {}^tU|},$$

where the surface integral is taken over the part of the surface $\bar{\alpha}\otimes\bar{t}\bar{\alpha}^{-1}$, satisfying the conditions given under the symbol of integration. The matrix U contains a unitary sub matrix, and by this reason we have $\left|U \cdot {}^tU\right| \geq 1$. In accordance with the Hadamard's inequality ([3, p. 154], [11]) we have the bound $\left|U \cdot {}^tU\right| \leq 2^{2j(r-1)}r^2H^{-2r}\lambda^{2r}G_1^2 = Y$. Make the linear transformation over the line manifold $\bar{\alpha} \otimes \bar{t}\bar{\alpha}^{-1}$, acting to it by the matrix T from the singular value decomposition for the matrix D_1. Since T is an orthogonal matrix after the transformation $\bar{u} = T\bar{x}$ view of the integral will not be changed. So, taking into account the above estimations we will have:

$$Y^{-1} \int\limits_{\|\Sigma\bar{u}\|\leq 1} d\sigma \leq \int\limits_{\|D_1(\bar{\alpha}\otimes \bar{t}\bar{\alpha}^{-1})\|\leq 1} dt_1 \cdots dt_r = \int\limits_{\|D_1\bar{x}\|\leq 1} \frac{ds}{|U \cdot {}^tU|} \leq \int\limits_{\|\Sigma\bar{u}\|\leq 1} d\sigma,$$

where $d\sigma$ means an element of the surface on the surface $\bar{u} = T(\bar{\alpha} \otimes \bar{t}\bar{\alpha}^{-1})$ (these estimations show that the surface integrals with respect ds and $d\sigma$ converge or diverge simultaneously). If this manifold has a dimension less than r then as a result of excess of variables, the surface integral diverges, i.e. $|J| = 0$. But this equality can be satisfied only on the set of points $\bar{\xi}'$ which set up a subset of zero measure. So, we can assume that the considered linear manifold has a dimension r. Denote the first r components of the vector $\bar{u}$ by $u_1, ..., u_r$. If the sub manifold generated by these variables has a dimension less r then we have the same situation considered above. So, we can assume that all of variables are independent and by this reason we must have:

$$\int\limits_{\|\Sigma\bar{u}\|} d\sigma = \int\limits_{\sigma_1u_1^2+\cdots+\sigma_ru_r^2\leq 1} d\bar{u} = c'\sigma_1^{-1}\cdots\sigma_r^{-1} = c'\det(D_1 \cdot {}^tD_1)^{-1/2} = c'\delta^{-1}. \quad (18)$$

(see [4, p.148]) (c' is a constant). Performing inverse transformation we find $T^{-1}\bar{u} = \bar{\alpha}\otimes\bar{t}\bar{\alpha}^{-1}$. Taking into account the bounds (16) of variation of variables, and integrating with respect to

α_j under the integral we get:

$$\int\limits_{\left(G_{(1)}\right)^{-1}r^{-1}2^{j(1-r)}H^rL^{-(1-r)^2}t_j}^{2^jH^{-1}L^{r-1}t_j} \frac{d\alpha_i}{\alpha_i} = \log(rG_{(1)}2^{jr}H^{-r-1}L^{r(r-1)}).$$

From the told above we conclude:

$$\int\limits_{\|\eta\|\le 1} d\bar{\eta} \int\limits_{\tau(\bar{\eta})} d\bar{\alpha} \int\limits_{\left\|D_1(\bar{\alpha}\otimes\bar{\beta})\right\|\le 1} d\bar{\beta} \le c_0\delta_1^{-1}\wp_j^r, \tag{19}$$

where $\wp_j = 1+\log(rG_{(1)}2^{jr}H^{-r-1}L^{r(r-1)})$. Therefore, using the estimation (9), we get the following bound for the integral over $K\backslash K_0$ on the right hand side of the equation (8):

$$\int\limits_{\bar{\alpha}\in K\backslash K_0} d\bar{\alpha} \int\limits_{\bar{\xi}'\in\tau,\|^tD_0\bar{\alpha}\|\le 1} d\bar{\xi}' \le c_0^2 H\left(\sum_{j=1}^{\infty}\wp_j^r 2^{1-j}\right)\delta_1^{-1}.$$

Designating $X = rG_{(1)}H^{-r-1}L^{r(r-1)}$ we consider the following sum:

$$\sum_{j=1}^{\infty}\wp_j^r 2^{1-j} = 2\sum_{j=1}^{\infty}[1+rj\log 2+\log X]^r 2^{-j}.$$

For estimation of this sum we notice that if $1+\log X > 2r^2$ then the function

$$\exp r\log(1+rj\log 2+\log X) - 0.5j\log 2.$$

monotonously decreases with respect to j. When $1+\log X \le 2r^2$ this function has the maximal value $\le 2^r r^{2r}$. Therefore,

$$\sum_{j\ge 0}[1+rj\log 2+\log X]^r 2^{-j/2}2^{-j/2} \le 2\left(1+r^2+\log\left(rG_{(1)}H^{-r-1}L^{r(r-2)}\right)\right)^r.$$

So,

$$\int\limits_{\Pi_1',|detD_0|\le H} d\xi_1\cdots d\xi_r \le 2^{r+3}r^{3r}c_0^2\left(C_n^r\right)^{1/2}H\delta_1^{-1}\wp^r. \tag{20}$$

$$\wp = 1+r^2+\log\left(rG_{(1)}H^{-r-1}L^{r(r-1)}\right).$$

Then,

$$\delta_1^2 \ge \left(C_m^r\right)^{-1}G_1(\bar{\xi}); \tag{21}$$
$$G_1(\bar{\xi}) = \det\left(A_1(\bar{\xi})\cdot {}^tA_1(\bar{\xi})\right).$$

Therefore,

$$\int\limits_{\Pi_1',|\det D_0|\le H} d\xi_1\cdots d\xi_r \le 2^{r+3}r^{3r}c_0^2\begin{pmatrix} n \\ r\end{pmatrix}^{1/2}\begin{pmatrix} rn \\ r\end{pmatrix}^{1/2} HG_{(1)}^{-1}\wp^r,$$

thus,

$$\wp = 1+r^2+\log\left(rG_{(1)}H^{-r-1}L^{r(r-1)}\right) \le 1+r^2+\log r+\log\{h(G_{(0)})h(H)^{r-1}h(L)^{r^2}\}$$

For $r\ge 2$ we have $h(a)\ge 2$. So,

$$r^2\log\left(h(G_{(1)})h(H)h(L)\right) \ge 3r^2+\log\{h(G_{(0)})h(H)^{r-1}h(L)^{r^2}\}.$$

The theorem 1 is proved. □

Corollary 3.1. *Let the conditions of the theorem satisfied. Then there exists a constant C such that*

$$\mu\left(\Pi_H\right) \leq CHG_1^{-1} \cdot \wp^r,$$

where

$$\wp = r^2 \log\left\{h\left(G_1\right) h\left(H\right) h\left(L\right)\right\}.$$

Proof. It is sufficient to prove that $G_{(1)} \geq G_1$. For any point $\bar{x}(\bar{\xi})$ on the surface we have

$$\det\left(A_{1\bar{\xi}} \cdot {}^t A_{1\bar{\xi}}\right)^{-1/2} =$$

$$= c_0 \int\limits_{\|{}^t A_{1\bar{\xi}}\bar{u}\| \leq 1} d\bar{u} = c_0' \int\limits_{\|{}^t A_1 \bar{x}\| \leq 1} ds,$$

where the surface integral is taken over tangential space to the surface at the point $\bar{x}(\bar{\xi})$. Considering the tangential space as a subspace of dimension r of the space R^n take the maximal value over all r-dimensional subspaces (see [4, p.148]). This maximal value is equal to G_1^{-1}. The proof of the corollary is finished. □

4. Conclusions

The question on estimation of areas in multidimensional domains arises in various branches of the mathematics. There are different ways for the solution of the problem in different cases. The method developed in this paper allows us to reduce the question to the investigation of operators defined by matrices given on tangential spaces on manifolds. In many questions such a reduction simplifies the estimation by using operators with a discrete spectrum.

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